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On Bitangential Interpolation in the Time-Varying Setting for Hilbert–Schmidt Operators: The Continuous Time Case

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The Hilbert space of lower triangular Hilbert–Schmidt operators on the real line is a natural analogue of the Hardy space of a half-plane, where the complex numbers are now replaced by matrix-valued functions. One can associate with a bounded operator its "values" at a matrix-valued function [see Ball *et al.*, *Oper. Theory Adv. Appl.* **56** (1992), 52–89], and this allows [see Ball *et al.*, *Integral Equations Operator Theory* **20** (1994), 1–43] to define and solve the analogue of the two-sided Nudelman interpolation problem for bounded operators (which form an analogue of $\mathbf{H}_\infty(\mathbb{C}_+)$). In this paper we consider the two-sided interpolation problem with a Hilbert–schmidt norm constraint (rather than the more common operator-norm constraint) on the interpolant. © 1998 Academic Press

1. INTRODUCTION

The bitangential interpolation problem (BIP for short; see [10] and [12] for the definition) in the classes of functions analytic and bounded in the open unit disk $\mathbb{D}$ or in the open upper half-plane $\mathbb{C}_+$ play an important role in the theory of time-invariant linear systems. This problem was studied in [2] and [3] in the setting of Hardy functions. The need to consider time-varying systems leads to natural generalizations of the notion of analytic functions in $\mathbb{D}$ or in $\mathbb{C}_+$: in the first case, analytic functions are replaced by bounded upper triangular operators between two sequence spaces while in the second case they are replaced by bounded lower triangular integral operators. The complex numbers are replaced,

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respectively, by diagonal operators and by multiplication operators by bounded measurable functions. In each case, there is a natural generalization of point evaluation, and this allows us to define the bitangential interpolation problem in the nonstationary setting. The case of upper triangular contractions (which corresponds to discrete time time-varying system) was considered in [4], [11], [6], [7], and [13]. The case of lower triangular contractive integral operators (which corresponds to the continuous time case) was studied in [5], [8], and [9].

The present paper is part of a series where the BIP is considered in Hardy spaces and in various analogues of these. In [1] solutions of the BIP which satisfy a preassigned bound on the Hilbert–Schmidt norm are sought. The objective of the present paper is to study this problem in the setting of lower triangular integral Hilbert–Schmidt operators. To make the statement of the problem precise we first recall some definitions and introduce the necessary notations.

The symbols $\mathbf{L}_\infty^{m\times n}(\mathbb{R})$, $\mathbf{L}_2^{m\times n}(\mathbb{R})$, and $\mathbf{L}_{\infty,2}^{m\times n}(\mathbb{R})$ stand for the set of all $m\times n$ matrices with entries in $\mathbf{L}_\infty(\mathbb{R})$, $\mathbf{L}_2(\mathbb{R})$, and $\mathbf{L}_{\infty,2}(\mathbb{R}) := \mathbf{L}_\infty(\mathbb{R}) \cap \mathbf{L}_2(\mathbb{R})$, respectively. Every function $A \in \mathbf{L}_\infty^{m\times n}(\mathbb{R})$ induces a multiplication operator $\mathbf{M}_A$ (which by abuse of notation will also be denoted by A) via the rule $(\mathbf{M}_A f)(t) = A(t)f(t)$.

Let Δ denote the operator of differentiation d/dt. The domain $D(\Delta)$ of the operator $\Delta = d/dt$ is the set of all functions $f \in \mathbf{L}_2(\mathbb{R})$ such that f is absolutely continuous on finite intervals and $\Delta f \in \mathbf{L}_2(\mathbb{R})$. The operator $\Delta - \mathbf{M}_A$ is a well-defined unbounded operator on $L_2^m(\mathbb{R})$ with domain $D(\Delta - \mathbf{M}_A) = D(\Delta)$. Let $U_A(t)$ be a matrix function satisfying the differential equation

$$\Delta U_A(t) = A(t)U_A(t), \qquad U_A(0) = I_m \qquad (t \in \mathbb{R}). \tag{1.1}$$

In other words, $U_A(t)$ is the fundamental matrix (normalized to the identity at $t = 0$) associated with the differential equation

$$\Delta x(t) = A(t)x(t), \qquad t \in \mathbb{R}. \tag{1.2}$$

Let us define the matrix function

$$\mathbf{U}_A(t,s) := U_A(t)U_A(s)^{-1}. \tag{1.3}$$

which (as is easily seen from (1.1)) solves the following Cauchy problem:

$$\frac{d}{dt}\mathbf{U}_A(t,s) = A(t)\mathbf{U}_A(t,s), \qquad \mathbf{U}_A(s,s) = I_n. \tag{1.4}$$

Note also that, in view of (1.1),

$$\frac{d}{dt}U_A^{-1} = -U_A^{-1}U_A'U_A^{-1} = -U_A^{-1}AU_AU_A^{-1} = -U_A^{-1}A$$

and, therefore,

$$\frac{d}{dt}\mathbf{U}_A(s,t) = -\mathbf{U}_A(s,t)A(t). \tag{1.5}$$

Following [5] and [8] we call A a backward stable time-variant point or stable time-variant point if there exist two constants $M > 0$ and a $(0 < a < 1)$ such that

$$\begin{aligned} &\|\mathbf{U}_A(t,s)\| \leq Ma^{s-t} \qquad (\text{for } s \geq t) \quad \text{or} \\ &\|\mathbf{U}_A(t,s)\| \leq Ma^{s-t}(\text{for } s \leq t), \end{aligned} \tag{1.6}$$

respectively. In these two cases the differential operator $\Delta - \mathbf{M}_A$ is boundedly invertible. We denote by $R(A)$ the operator acting via the rule

$$(R(A)\varphi)(t) = \left((\Delta - \mathbf{M}_A)^{-1}\varphi\right)(t), \qquad \varphi \in \mathbf{L}_2^m(\mathbb{R}),$$

which can be written as follows

$$(R(A)\varphi)(t) = -\int_t^{\infty} \mathbf{U}_A(t,s)\varphi(s)\, ds$$

or

$$(R(A)\varphi)(t) = \int_t^{\infty} \mathbf{U}_A(t,s)\varphi(s)\, ds \qquad (t \in \mathbb{R}),$$

whenever A is a backward stable time-variant point or stable time-variant point, respectively.

We denote by $\mathscr{X}_2^{m\times n}$ the class of Hilbert−Schmidt integral operators $F = F_k$ generated by $\mathbf{L}_2^{m\times n}(\mathbb{R}^2)$-functions $k(t,s)$ and acting from $\mathbf{L}_2^n$ into $\mathbf{L}_2^m$ by the rule

$$(F\varphi)(t) = -\int_{-\infty}^{+\infty} k(t,s)\varphi(s)\, ds, \qquad \varphi \in \mathbf{L}_2^m(\mathbb{R}).$$

The space of Hilbert−Schmidt operators endowed with the inner product

$$\langle H, H\rangle_{\chi_2} \overset{\text{def}}{=} \operatorname{Tr}(H^*H)$$

is a Hilbert space. We say that the operator $F_k \in \mathscr{X}_2^{m\times n}$ is anticausal (causal) if the corresponding kernel function $k(t,s)$ satisfies the condition $k(t,s) = 0$ for $s \leq t$ ($k(t,s) = 0$ for $s \geq t$). We denote the spaces of all anticausal and causal operators by $\mathscr{U}_2^{m\times n}$ and $\mathscr{L}_2^{m\times n}$, respectively, and refer to these spaces as to the spaces of upper (respectively, lower) triangular Hilbert–Schmidt operators.

We remark that in this setting the class of lower (upper) triangular operators plays the role of the class of functions analytic in the upper (lower) half-plane whereas multiplication operators play the role of constants. It is easily seen that $\mathscr{X}_2 = \mathscr{U}_2 \oplus \mathscr{L}_2$ and we denote by $\underline{\underline{q}}$ and $\underline{\underline{p}}$ the orthogonal projections of $\mathscr{X}_2$ onto $\mathscr{U}_2$ and $\mathscr{L}_2$, respectively.

It was shown in [5, Lemma 1.1] that, for any backward stable (respectively, stable) time-variant point A, the operator $(\Delta - \mathbf{M}_A)^{-1}\mathbf{M}_\varphi)$, where $\varphi \in \mathbf{L}_2^m$, is a Hilbert–Schmidt operator. Moreover, by definition, this operator is anticausal (respectively, causal), i.e., belongs to $\mathscr{U}_2$ (respectively, to $\mathscr{L}_2$).

To formulate the interpolation problem we need to define the "point" evaluation for lower triangular Hilbert–Schmidt operators. The next lemma establishes the existence and uniqueness of the left and right point evaluations, and is a nonstationary analogue of Cauchy's formula. For the proof, see [9, Propositions 2.2 and 2.3] and [8, Proposition 2.2]. For the discrete analogue, see [4, Theorem 7.3, p. 106].

LEMMA 1.1. *Let $F \in \mathscr{L}_2^{m\times n}$, and let $A_\zeta \in L_\infty^{m\times m}(\mathbb{R})$ and $A_\pi \in L_\infty^{n\times n}(\mathbb{R})$ be backward stable time-variant points. Then there exist two functions $X \in L_2^{m\times n}(\mathbb{R})$, $Y \in L_2^{m\times n}(\mathbb{R})$ such that*

$$\underline{\underline{q}}(F - X)R(A_\pi) = 0 \quad \textit{and} \quad \underline{\underline{q}}R(A_\zeta)(F - Y) = 0.$$

The operators X and Y are called *the generalized point evaluations* and are denoted by $X = F^{\wedge_R}(A_\pi)$ and $Y = F^{\wedge_L}(A_\zeta)$; they can be written explicitly in terms of the kernel function $k(t,s)$ of the operator $F = F_k$; see [5, Proposition 1.1].

We consider the interpolation problem whose data set is an ordered collection

$$\Omega = \{C_+, C_-, A_\pi, A_\zeta, B_+, B_-, \Gamma\} \tag{1.7}$$

of matrix-valued functions

$$C_+ \in \mathbf{L}_{\infty,2}^{n\times n_\pi}(\mathbb{R}), \qquad C_- \in \mathbf{L}_\infty^{m\times n_\pi}(\mathbb{R}),$$
$$B_+ \in \mathbf{L}_\infty^{n_\zeta\times n}(\mathbb{R}), \qquad B_- \in \mathbf{L}_{\infty,2}^{n_\zeta\times m}(\mathbb{R}),$$

of backward stable time-varying points $A_\pi \in \mathbf{L}_\infty^{n_\pi \times n_\pi}(\mathbb{R})$ and $A_\zeta \in \mathbf{L}_\infty^{n_\zeta \times n_\zeta}(\mathbb{R})$ and of a matrix-valued function $\Gamma \in \mathbf{L}_{\infty,2}^{n_\zeta \times n_\pi}(\mathbb{R})$, which is absolutely continuous with respect to Lebesgue measure and satisfies the following differential equation:

$$\frac{d}{dt}\Gamma(t) = A_\zeta(t)\Gamma(t) - \Gamma(t)A_\pi(t) + B_-(t)C_-(t) - B_+(t)C_+(t). \tag{1.8}$$

The problem associated with the data set (1.7) is:

Problem 1.2. Find all operators $H \in \mathscr{L}_2^{m\times n}(\mathbb{R})$ such that

$$\underline{\underline{q}}(HC_- - C_+)R(A_\pi) = 0, \tag{1.9}$$

$$\underline{\underline{q}}R(A_\zeta)(B_+H - B_-) = 0, \tag{1.10}$$

$$\underline{\underline{q}}R(A_\zeta)\{B_+(HC_- - C_+)R(A_\pi) + \Gamma\} = 0 \tag{1.11}$$

under the norm constraint

$$\|H\|_{\chi_2}^2 \overset{\text{def}}{=} \operatorname{Tr}(H^*H) \le 1. \tag{1.12}$$

Note that condition (1.9) prescribes the right generalized value of HC_- at the backward stable time-varying point A_π and is therefore *the right-sided condition* with respect to the interpolant H. Similarly, condition (1.10) is called *the left-sided condition*, since it preassigns the left generalized value of B_+H at a backward stable time-varying point A_ζ and is therefore left-sided with respect to the interpolant H. The third condition (1.11) has to be added in order to take care of the compatibility of the previous two.

The variation of Problem 1.2 where condition (1.12) is replaced by

$$\|h\|_\infty = \sup_{h \in \mathscr{L}_2^n(\mathbb{R})} \|Hh\| \le 1$$

is the topic of the papers [5], [8], and [9]. There it is shown that solutions exist if and only if a certain time-varying Pick matrix $\Lambda(t)$ is positive semidefinite for all real t, and, in the nondegenerate case where the Pick matrix is uniformly strictly positive definite, the set of all solutions H can be parametrized via a linear fractional transformation

$$H = (\Theta_{11}h + \Theta_{12})(\Theta_{21}h + \Theta_{22})^{-1},$$

where the coefficients Θ_{ij} are ultimately determined by the interpolation data, and h is a free-parameter lower triangular integral operator with

$\|h\|_\infty \leq 1$. In this paper we obtain the analogous existence criterion and parametrization of the set of all solutions for Problem 1.2. The set of all solutions is given in the affine form

$$H = H_{\min} + \Theta_L h \Theta_R,$$

where $H_{\min}$, Θ_L, and Θ_R are explicitly computable from the interpolation data, and h is a free-parameter lower triangular integral operator with $\|h\|_2^2 \leq 1 - \|H_{\min}\|_2^2$. It will be shown that $H_{\min}$ is the minimal Hilbert–Schmidt norm solution and therefore the existence criterion of the problem is that $\|H_{\min}\|_2 \leq 1$.

The outline of the paper is as follows: the paper consists of four sections besides the Introduction. In the second section we review the continuous setting developed in [5], [8], and [9]. The right-sided interpolation problem (i.e., with only the interpolation condition (1.9) is solved in Section 3. The dual results concerning the left-sided problem (i.e., with only condition (1.10)) are stated in Section 4. Finally, in Section 5 we show that the two-sided interpolation problem can be reduced to a pair of one-sided problems and obtain the parametrization of all its solutions using the preceding analysis.

2. PRELIMINARIES

In this section we present a number of preliminary lemmas which will be needed in the sequel.

LEMMA 2.1. *Let $\Gamma \in L_\infty^{n\times m}(\mathbb{R})$ be absolutely continuous with respect to Lebesgue measure and let $A_\pi \in L_\infty^{n\times n}(\mathbb{R})$ and $A_\zeta \in L_\infty^{m\times m}(\mathbb{R})$ be backward stale time-variant points. Then Γ is a solution of the Lyapunov equation*

$$\Gamma R(A_\pi) - R(A_\zeta)\Gamma = R(A_\zeta)(B_- C_- - B_+ C_+) R(A_\pi) \qquad (2.1)$$

if and only if Γ satisfies the differential equation (1.8).

Proof. For the proof of necessity, see [9, Sect. 4]. To prove sufficiency, we take $\psi \in L_2^n(\mathbb{R})$ and set $\phi = R(A_\pi)\psi$, which belongs to $L_2^n(\mathbb{R})$. Moreover, ϕ will be absolutely continuous with respect to Lebesgue measure for ψ in a dense set of $L_2^n(\mathbb{R})$. Applying both sides of (2.1) to the function ψ, we obtain the equation

$$\Gamma\phi - R(A_\zeta)\Gamma\psi = R(A_\zeta)(B_- C_- - B_+ C_+)\phi.$$

Applying the operator $(\Delta - A_\zeta)$ to both sides of the latter equation, we get

$$(\Delta - A_\zeta)\Gamma \quad \Gamma(\Delta - A_\pi)\phi = (B_- C_- - B_+ C_+)\phi.$$

This implies (1.8), since

$$\Delta\Gamma\phi - \Gamma\Delta\phi = \Gamma'\phi$$

for every choice of $\Gamma \in L_\infty^{n\times m}(\mathbb{R})$ and $\phi \in L_2^n(\mathbb{R})$, which are absolutely continuous with respect to Lebesgue measure. ∎

The next lemma can be proved using the arguments from the proof of Lemma 2.1; details are omitted.

LEMMA 2.2. *Let $P \in \mathbf{L}_\infty^{n\times n}(\mathbb{R})$ be absolutely continuous on any finite interval of the real axis $\mathbb{R}$, let B and C be any elements of $\mathbf{L}_\infty^{m\times n}(\mathbb{R})$ and $\mathbf{L}_\infty^{n\times m}(\mathbb{R})$, respectively, and let $A \in L_\infty^{n\times n}(\mathbb{R})$ be a backward stable time-variant point. Then*

1. *P is a solution of the Lyapunov equation*

$$R(A)BB^*R(A)^* = R(A)P + PR(A)^* \tag{2.2}$$

if and only if

$$\frac{d}{dt}P = AP + PA^* + BB^*; \tag{2.3}$$

2. *P is a solution of the Lyapunov equation*

$$R(A)^*C^*CR(A) = R(A)^*P + PR(A) \tag{2.4}$$

if and only if

$$-\frac{d}{dt}P = A^*P + PA + C^*C. \tag{2.5}$$

LEMMA 2.3. *Assume that both P and P^{-1} are bounded matrix functions which are absolutely continuous on any finite interval of $\mathbb{R}$, and let $A \in L_\infty^{n\times n}(\mathbb{R})$ be a backward stable time-variant point. Then P is a solution of the Lyapunov equation* (2.4) (*or, equivalently, of* (2.5)) *if and only if P^{-1} is a solution of the Lyapunov equation*

$$R(A)^*P^{-1}C^*CP^{-1}R(A) = R(A)^*P^{-1} + P^{-1}R(A) \tag{2.6}$$

or, equivalently, if and only if

$$\frac{d}{dt}P^{-1} = P^{-1}A^* + AP^{-1} + P^{-1}C^*CP^{-1}. \tag{2.7}$$

Proof. Upon replacing in (2.4) and (2.5) P and C by P^{-1} and CP^{-1}, respectively, we conclude by Lemma 2.2 that conditions (2.6) and (2.7) are equivalent. To complete the proof it suffices to show that (2.7) is equivalent to (2.5). But this is self-evident: upon multiplying both sides of (2.5) by P^{-1} on the left and on the right and taking into account that $(d/dt)P^{-1} = -P^{-1}P'P^{-1}$ we get (2.5). ∎

It turns out that the Lyapunov equations (2.2) and (2.4) have unique bounded solutions. The next lemma expresses these solutions in terms of $\mathbf{U}_A(t,s)$, the solution of the Cauchy problem (1.4).

LEMMA 2.4. *The Lyapunov equations* (2.2) *and* (2.4) *always have positive semidefinite solutions*:

$$F_1(t) = \int_{-\infty}^{t} \mathbf{U}_A(t,s)B(s)B(s)^*\mathbf{U}_A(t,s)^*\,ds \qquad (t \in \mathbb{R})$$

is a solution of (2.2) *and*

$$F_2(t) = \int_{t}^{\infty} \mathbf{U}_A(s,t)^*C(s)^*C(s)\mathbf{U}_A(s,t)\,ds \qquad (t \in \mathbb{R}) \tag{2.8}$$

is a solution of (2.4). *Moreover, these solutions F_1 and F_2 are the unique bounded solutions of the equations* (2.2) *and* (2.4), *respectively*.

Proof. Upon differentiating (2.8) and using (1.5) we get

$$\begin{aligned} F_2'(t) &= -\mathbf{U}_A(t,t)^*C(t)^*C(t)\mathbf{U}_A(t,t) \\ &\quad - A(t)^*\int_t^{\infty}\mathbf{U}_A(s,t)^*C(s)^*C(s)\mathbf{U}_A(s,t)\,ds \\ &\quad - \int_t^{\infty}\mathbf{U}_A(s,t)^*C(s)^*C(s)\mathbf{U}_A(s,t)\,ds\,A(t) \\ &= -C(t)^*C(t) - A(t)^*F_2(t) - F_2(t)A(t), \end{aligned}$$

which means that the function $P = F_2$ satisfies (2.5) (which, in turn, is equivalent to (2.4), by Lemma 2.2). To show the uniqueness, let P be a solution of (2.5). Then, in view of (1.4),

$$\begin{aligned}
&\frac{d}{dt}\langle P(t)\mathbf{U}_A(t,s)x_0(s), \mathbf{U}_A(t,s)x_0(s)\rangle \\
&\qquad = \langle \mathbf{U}_A(t,s)^*(P'(t) + P(t)A(t) + A(t)^*P(t)) \\
&\qquad\qquad \times \mathbf{U}_A(t,s)x_0(s), x_0(s)\rangle \\
&\qquad = -\langle \mathbf{U}_A(t,s)^*C(t)^*C(t)\mathbf{U}_A(t,s)x_0(s), x_0(s)\rangle.
\end{aligned}$$

Since A is a backward stable time-varying point, (1.6) holds and the integral

$$\begin{aligned}
&\int_s^\infty \frac{d}{dt}\langle P(t)\mathbf{U}_A(t,s)x_0(s), \mathbf{U}_A(t,s)x_0(s)\rangle\, dt \\
&\qquad = -\int_s^\infty \langle \mathbf{U}_A(t,s)^*C(t)^*C(t)\mathbf{U}_A(t,s)x_0(s), x_0(s)\rangle\, dt
\end{aligned}$$

converges, since $C(t)^*C(t) \in L_\infty^{n\times n}(\mathbb{R})$. Hence the latter integral is equal to

$$-\langle P(s)\mathbf{U}_A(s,s)x_0(s), \mathbf{U}_A(s,s)x_0(s)\rangle = -\langle P(s)x_0(s), x_0(s)\rangle.$$

On the other hand, in view of (2.8),

$$\int_s^\infty \langle \mathbf{U}_A(t,s)^*C(t)^*C(t)\mathbf{U}_A(t,s)x_0(s), x_0(s)\rangle\, dt = \langle F_2(s)x_0(s), x_0(s)\rangle$$

and, since x_0 is arbitrary, $F_2(s) = P(s)$ for every $s \in \mathbb{R}$. To prove that F_1 is the unique solution of (2.2) one can use in much the same arguments. ∎

We recall that from the system-theoretic point of view the operators F_1 and F_2 are observability and controllability Gramians (see [5] and [8] for more discussion on the subject).

3. RIGHT-SIDED INTERPOLATION

In this section we consider the following right-sided interpolation problem: *find all operators* $H \in \mathscr{L}_2^{m\times n}(\mathbb{R})$ *satisfying the right-sided interpolation condition* (1.9).

DEFINITION. We shall call an operator M_P a strictly positive operator if its symbol $P(t)$ is strictly positive, i.e., $P(t) \geq \varepsilon I$ almost everywhere on $\mathbb{R}$, for some $\varepsilon > 0$.

It is perhaps worth mentioning that if M_P is strictly positive then there exists $M_P^{-1} = M_{P^{-1}}$.

LEMMA 3.1. *Let $\mathbb{P}_R$ be a strictly positive bounded solution of the Lyapunov equation associated with the interpolation problem under study*:

$$R(A_\pi)^* C_-^* C_- R(A_\pi) = R(A_\pi)^* \mathbb{P}_R + \mathbb{P}_R R(A_\pi). \tag{3.1}$$

Then the operator

$$H_R = C_+ \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* \tag{3.2}$$

belongs to $\mathscr{L}_2^{m\times n}(\mathbb{R})$ and satisfies (1.9).

Proof. Upon using formula (3.2) we can write

$$(H_R C_- - C_+) R(A_\pi) = C_+ \mathbb{P}_R^{-1} \{ R(A_\pi)^* C_-^* C_- R(A_\pi) - \mathbb{P}_R R(A_\pi) \}.$$

Then using (3.1) we obtain that

$$(H_R C_- - C_+) R(A_\pi) = C_+ \mathbb{P}_R^{-1} R(A_\pi)^* \mathbb{P}_R. \tag{3.3}$$

Since the operators C_+ $\mathbb{P}_R$, and $\mathbb{P}_R^{-1} \in \mathscr{D}$ and $R(A_\pi)^*$ is lower triangular, we conclude that $\underline{\underline{q}}(HC_- - C_+)R(A_\pi) = 0$. ∎

LEMMA 3.2. *Let $\mathbb{P}_R$ be a boundedly invertible operator which satisfies* (3.1). *Then the operator*

$$\Theta_R = I - C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* \in \mathscr{L}^{n\times n} \tag{3.4}$$

is lower triangular and unitary.

Proof. The fact that Θ_R is lower triangular is obvious since all the terms in (3.4) are lower triangular operators. To prove that Θ_R is unitary we use (3.1):

$$\begin{aligned} I - \Theta_R \Theta_R^* &= C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* + C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^* \\ &\quad - C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^* \\ &= C_- \mathbb{P}_R^{-1} \{ R(A_\pi)^* \mathbb{P}_R + \mathbb{P}_R R(A_\pi) \\ &\qquad - R(A_\pi)^* C_-^* C_- R(A_\pi) \} \mathbb{P}_R^{-1} C_-^* = 0. \end{aligned}$$

Since $\mathbb{P}_R$ is the boundedly invertible solution of the Lyapunov equation (3.1), by Lemma 2.3, we conclude that

$$R(A_\pi) \mathbb{P}_R^{-1} C_-^* C_- \mathbb{P}_R^{-1} R(A_\pi)^* = \mathbb{P}_R^{-1} R(A_\pi)^* + R(A_\pi) \mathbb{P}_R^{-1}. \tag{3.5}$$

Therefore,

$$\begin{aligned} I - \Theta_R^* \Theta_R &= C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* + C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^* \\ &\quad - C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^* C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^* \\ &= C_- \big\{ \mathbb{P}_R^{-1} R(A_\pi)^* + R(A_\pi) \mathbb{P}_R^{-1} \\ &\qquad - R(A_\pi) \mathbb{P}_R^{-1} C_-^* C_- \mathbb{P}_R^{-1} R(A_\pi)^* \big\} C_-^* = 0, \end{aligned}$$

which completes the proof of the lemma. ∎

THEOREM 3.3. *An operator $H \in \mathscr{L}_2^{m \times n}$ satisfies* (1.9) *if and only if it can be represented as*

$$H = H_R + \hat{H} \Theta_R, \tag{3.6}$$

where H_R and Θ_R are given by (3.2) *and* (3.4), *respectively, and $\hat{H}$ is an element from $\mathscr{L}_2^{m \times n}$.*

Proof. Let $H \in \mathscr{L}_2^{m \times n}$ satisfy (1.9). Then, according to Lemma 3.1, the operator $S := H - H_R$ is subject to the right-sided homogeneous condition

$$\underline{\underline{q}} S C_- R(A_\pi) = 0. \tag{3.7}$$

To complete the proof, it suffices to prove that an operator $S \in \mathscr{L}_2^{m \times n}$ satisfies the homogeneous condition (3.7) if and only if it is of the form

$$S = \hat{H} \Theta_R \qquad \text{for some } \hat{H} \in \mathscr{L}_2^{m \times n}. \tag{3.8}$$

Let $S \in \mathscr{L}_2^{m \times n}$ satisfy (3.7). Upon setting $\hat{H} := S \Theta_R^*$, we get

$$\hat{H} = S \Theta_R^* = S\big(I - C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^*\big).$$

Since $S \in \mathscr{L}_2$, $\mathbb{P}_R^{-1} \in \mathscr{D}$, $C_-^* \in \mathscr{D}_2$, and, on account of (3.7),

$$\underline{\underline{q}} S C_- R(A_\pi) \mathbb{P}_R^{-1} C_-^* = 0$$

and, therefore, $\hat{H} \in \mathscr{L}_2^{m \times n}$.

To prove the converse we note that in view of (3.1) and (3.4),

$$\begin{aligned} \Theta_R C_- R(A_\pi) &= \big(I - C_- \mathbb{P}_R^{-1} R(A_\pi)^* C_-^*\big) C_- R(A_\pi) \\ &= C_- R(A_\pi) - C_- \mathbb{P}_R^{-1} \{ R(A_\pi)^* \mathbb{P}_R + \mathbb{P}_R R(A_\pi) \} \\ &= -C_- \mathbb{P}_R^{-1} R(A_\pi)^* \mathbb{P}_R, \end{aligned} \tag{3.9}$$

which implies that the operator $\Theta_R C_- R(A_\pi)$ is lower triangular. Therefore, for every $\hat{H} \in \mathscr{L}_2^{m\times n}$

$$\hat{H}\Theta_R C_- R(A_\pi) = SC_- R(A_\pi) \in \mathscr{L}_2^{m\times n},$$

which implies (3.7) and completes the proof of the theorem. ▮

LEMMA 3.4. *The operator H_R given by* (3.2) *has minimal norm among all $\mathscr{L}_2^{m\times n}$-operators satisfying the interpolation condition* (1.9).

Proof. To prove this lemma it is enough to show that H_R is orthogonal to $\hat{H}\Theta_R$. Indeed, according to Theorem 3.3 every $H \in \mathscr{L}_2^{m\times n}$ satisfying (1.9) admits the representation (3.6). Therefore, $\|H_R\| \leq \|H\|$. The proof that H_R is orthogonal to $\hat{H}\Theta_R$ is a straightforward calculation which is based on the formulas (3.2) and (3.4). Indeed, from these formulas and the Lyapunov equation (3.1) it follows that

$$H_R \Theta_R^* = C_+ R(A_\pi)\mathbb{P}_R^{-1} C_-^* \in \mathscr{L}_2^{m\times n}.$$

Since Θ_R is unitary, the statement of the lemma drops easily from the observation that

$$\langle H_R, \hat{H}\Theta_R \rangle_{\mathscr{X}_2} = \langle H_R \Theta_R^*, \hat{H} \rangle_{\mathscr{X}_2} = 0.$$

▮

As a simple consequence of the previous result one can describe the set of all the solutions to the right-sided interpolation problem with norm constraint.

THEOREM 3.5. *Let $\mathbb{P}_R$ be a strictly positive solution of the Stein equation* (3.1). *Then there exists a solution of the right-sided interpolation problem* (1.9) *of norm less than* 1 *if and only if* $\|\hat{H}_R\| \leq 1$.

Moreover, all the solutions of norm less than 1 *can be described via formula* (3.6), *where in this case the parameter $\hat{H} \in \mathscr{L}_2^{m\times n}$ satisfies the following norm constraint*:

$$\|\hat{H}\|_2 \leq \left(1 - \|H_R\|_2^2\right)^{1/2}.$$

4. LEFT-SIDED INTERPOLATION

In this section we consider the left-sided interpolation problem: *find all operators $H \in \mathscr{L}_2^{m\times n}(\mathbb{R})$ satisfying the left-sided interpolation condition* (1.10).

We supply here the facts concerning this problem. The proofs can be carried out in complete analogy to the proofs of the corresponding results of the previous section.

LEMMA 4.1. *Let P_L be a strictly positive solution of the Lyapunov equation* (2.2) *associated with the interpolation problem*:

$$R(A_\zeta)B_+B_+^*R(A_\zeta)^* = R(A_\zeta)\mathbb{P}_L + \mathbb{P}_L R(A_\zeta)^*.$$

Then the operator

$$H_L = B_+^* R(A_\zeta)\mathbb{P}_L^{-1}B_- \tag{4.1}$$

satisfies (1.10).

LEMMA 4.2. *Let $\mathbb{P}_L$ be a boundedly invertible operator which satisfies* (2.2). *Then*

$$\Theta_L = \left(I - B_+^* R(A_\zeta)^* \mathbb{P}_L^{-1} B_+\right) \tag{4.2}$$

is a lower triangular unitary operator.

THEOREM 4.3. *$H \in \mathscr{L}_2^{m\times n}$ satisfies the interpolation condition* (1.10) *if and only if it can be represented as*

$$H = H_L + \Theta_L \hat{H}, \tag{4.3}$$

where H_L and Θ_L are given by (4.1) *and* (4.2), *respectively, and $\hat{H} \in \mathscr{L}_2^{m\times n}$.*

LEMMA 4.4. *The solution H_L given by formula* (4.1) *has the smallest norm among all the solutions to the left-sided interpolation problem* (1.10).

Once again as a simple consequence of the previous lemma we can describe the set of all the solutions to the left-sided interpolation problem with norm constraint.

THEOREM 4.5. *Let $\mathbb{P}_L$ be a strictly positive solution of the Stein equation* (2.2). *Then there exists a solution of the right-sided interpolation problem* (1.10) *of norm less than* 1 *if and only if $\|\hat{H}_L\| \leq 1$.*

Moreover, all the solutions of norm less than 1 *can be described via formula* (4.3), *where in this case a parameter $\hat{H} \in \mathscr{L}_2^{m\times n}$ satisfies the following norm constraint,*

$$\|\hat{H}\|_2 \leq \left(1 - \|H_L\|_2^2\right)^{1/2}.$$

5. TWO-SIDED INTERPOLATION

Using the results from the previous sections we can describe the set of all the solutions to the two-sided problem. To do this we reduce Problem 1.2 to a pair of one-sided problems. Such a reduction is provided by the following result:

THEOREM 5.1. *Let* $H \in \mathscr{L}_2^{m\times n}(\mathbb{R})$ *be an operator satisfying* (1.9), *i.e., of the form* (3.6):

$$H = H_R + \hat{H}\Theta_R,$$

where H_R *and* Θ_R *are given by* (3.2) *and* (3.4), *respectively, and* $\hat{H}$ *is an element of* $\mathscr{L}_2^{m\times n}(\mathbb{R})$. *Then* H *satisfies also conditions* (1.10) *and* (1.11) *if and only if* $\hat{H}$ *is subject to the left-sided interpolation condition*

$$\underline{\underline{q}}R(A_\zeta)\left(B_+\hat{H} - \hat{B}_-\right) = 0, \tag{5.1}$$

where

$$\hat{B}_- = B_- + \Gamma\mathbb{P}_R^{-1}C_-^*. \tag{5.2}$$

Proof. First, we prove the sufficiency part. Let H be of the form (3.6) and let (5.1) be in force. Substituting (3.6) into (1.10) and (1.11) we conclude that it is enough to check that

$$\underline{\underline{q}}R(A_\zeta)\left(B_+H_R + B_+\hat{H}\Theta_R - B_-\right) = 0 \tag{5.3}$$

and

$$\underline{\underline{q}}R(A_\zeta)\left(B_+\left\{H_R + \hat{H}\Theta_R\right\}C_-R(A_\pi) - B_+C_+R(A_\pi) + \Gamma\right) = 0. \tag{5.4}$$

Since $\Theta_R \in \mathscr{L}^{n\times n}$ and in view of (5.1),

$$\underline{\underline{q}}R(A_\zeta)B_+\hat{H}\Theta_R = \underline{\underline{q}}\left(\left(\underline{\underline{q}}R(A_\zeta)B_+\hat{H}\right)\Theta_R\right) = \underline{\underline{q}}R(A_\zeta)\hat{B}_-\Theta_R,$$

which allows us to rewrite (5.3) in the following equivalent form:

$$\underline{\underline{q}}R(A_\zeta)\left(B_+H_R + \hat{B}_-\Theta_R - B_-\right) = 0. \tag{5.5}$$

Upon making use of the explicit formulas (3.2), (3.4), and (5.2) for H_R, Θ_R, and $\hat{B}_-$ we represent the left-hand side of the latter equality as

$$\begin{aligned}\underline{\underline{q}}R(A_\zeta)\{&(B_+C_+ - B_-C_-)\mathbb{P}_R^{-1}R(A_\pi)^*C_-^* + \Gamma\mathbb{P}_R^{-1}C_-^* \\ &-\Gamma\mathbb{P}_R^{-1}C_-^*C_-\mathbb{P}_R^{-1}R(A_\pi)^*C_-^*\} \\ = \underline{\underline{q}}\{&① + ② + ③\}.\end{aligned}$$

Since Γ satisfies the Lyapunov equation (2.1),

$$\begin{aligned} ① &= \left\{R(A_\zeta)\Gamma - \Gamma R(A_\pi)\right\}R(A_\pi)^{-1}\mathbb{P}_R^{-1}R(A_\pi)^* C_-^* \\ &= R(A_\zeta)\Gamma R^{-1}(A_\pi)\mathbb{P}_R^{-1}R(A_\pi)^* C_-^* - \Gamma\mathbb{P}_R^{-1}R(A_\pi)^* C_-^*, \end{aligned}$$

whereas usage of (3.5) leads to

$$\begin{aligned} ③ &= -R(A_\zeta)\Gamma R(A_\pi)^{-1}\left\{\mathbb{P}_R^{-1}R(A_\pi)^* + R(A_\pi)\mathbb{P}_R^{-1}\right\}C_-^* \\ &= -② - R(A_\zeta)\Gamma R(A_\pi)^{-1}\mathbb{P}_R^{-1}R(A_\pi)^* C_-^*. \end{aligned}$$

Hence,

$$\begin{aligned} \underline{\underline{q}}R(A_\xi)\left(B_+H_R + B_+\hat{H}\Theta_R - B_-\right) &= \underline{\underline{q}}\left\{① + ② + ③\right\} \\ &= -\underline{\underline{q}}\Gamma\mathbb{P}_R^{-1}R(A_\pi)^* C_-^* \end{aligned}$$

and, since $\Gamma\mathbb{P}_R^{-1}R(A_\pi)^* C_-^* \in \mathscr{L}_2^{m_\zeta \times m}$, the latter equality implies (5.3).

Next, in view of (3.9) and (3.3), condition (5.4) is equivalent to

$$\underline{\underline{q}}R(A_\zeta)\left(B_+C_+\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R - B_+\hat{H}C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R + \Gamma\right) = 0. \tag{5.6}$$

Furthermore, since $C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R \in \mathscr{L}^{n\times n_\pi}$ and in view of (5.1),

$$\begin{aligned} \underline{\underline{q}}R(A_\zeta)B_+\hat{H}C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R &= \underline{\underline{q}}\left(\left(\underline{\underline{q}}R(A_\zeta)B_+\hat{H}\right)C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R\right) \\ &= \underline{\underline{q}}R(A_\zeta)\hat{B}_-C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R, \end{aligned}$$

which allows us to represent (5.6) in the following equivalent form:

$$\underline{\underline{q}}R(A_\zeta)\left(B_+C_+\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R - \hat{B}_-C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R + \Gamma\right) = 0.$$

Upon using (5.2) we represent the left-hand side of the latter equality as

$$\begin{aligned} &\underline{\underline{q}}R(A_\zeta)\{(B_+C_+ - B_-C_-)\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R \\ &\qquad -\Gamma\mathbb{P}_R^{-1}C_+^*C_-\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R + \Gamma\} = \underline{\underline{q}}\left\{① + ② + ③\right\}. \end{aligned}$$

In view of (2.1),

$$\text{①} = R(A_\zeta)\Gamma R(A_\pi)^{-1}\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R - \Gamma\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R,$$

whereas (3.5) implies

$$\begin{aligned}\text{②} &= -R(A_\zeta)\Gamma - R(A_\zeta)\Gamma R(A_\pi)^{-1}\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R\\ &= -\text{③} - R(A_\zeta)\Gamma R(A_\pi)^{-1}\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R.\end{aligned}$$

Therefore,

$$\begin{aligned}&\underline{\underline{q}}R(A_\zeta)\big(B_+\{H_R + \hat{H}\Theta_R\}C_-R(A_\pi) - B_+C_+R(A_\pi) + \Gamma\big)\\ &\qquad = \underline{\underline{q}}\{\text{①} + \text{②} + \text{③}\}\\ &\qquad = -\underline{\underline{q}}\Gamma\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R\end{aligned}$$

and, since $\Gamma\mathbb{P}_R^{-1}R(A_\pi)^*\mathbb{P}_R \in \mathscr{L}^{n_\zeta\times n_\pi}$, the latter equality implies (5.4). This completes the proof of this part.

Now we prove the necessity part of the theorem: Let H be of the form (3.6) and satisfy interpolation conditions (1.10) and (1.11). Since Θ_R is unitary, it follows from (3.6) that

$$\hat{H} = (H - H_R)\Theta_R^*$$

and it remains to check that this $\hat{H}$ satisfies the interpolation condition (5.1):

$$\underline{\underline{q}}R(A_\zeta)\big(B_+(H - H_R)\Theta_R^* - \hat{B}_-\big) = 0. \tag{5.7}$$

Since H satisfies (1.10) and (1.11),

$$\begin{aligned}\underline{\underline{q}}R(A_\zeta)B_+H\Theta_R^* &= \underline{\underline{q}}R(A_\zeta)B_+H - \underline{\underline{q}}R(A_\zeta)B_+HC_-R(A_\pi)\mathbb{P}_R^{-1}C_-^*\\ &= \underline{\underline{q}}R(A_\zeta)B_- - \underline{\underline{q}}R(A_\zeta)B_+C_+R(A_\pi)\mathbb{P}_R^{-1}C_-^*\\ &\qquad + \underline{\underline{q}}R(A_\zeta)\Gamma\mathbb{P}_R^{-1}C_-^*.\end{aligned}$$

Next, from (3.2) and (3.4) using the Lyapunov equation (3.5) we have that

$$H_R\Theta_R^* = -C_+R(A_\pi)\mathbb{P}_R^{-1}C_-^*.$$

From the two latter equalities we get

$$\underline{\underline{q}}R(A_\zeta)B_+(H - H_R)\Theta_R^* = \underline{\underline{q}}R(A_\zeta)\big(B_- + \Gamma\mathbb{P}_R^{-1}C_-^*\big),$$

which is equivalent to (5.7) on account of (5.2). ▮

By Theorem 4.3, all $\hat{H} \in \mathscr{L}^{m\times n}$ satisfying condition (5.1) are parametrized by the formula

$$\hat{H} = \hat{H}_L + \Theta_L h, \tag{5.8}$$

where Θ_L is given by (4.2),

$$H_L = B_+^* R(A_\zeta) P_L^{-1} \hat{B}_- \tag{5.9}$$

and h is an arbitrary element of $\mathscr{L}^{m\times n}$. Therefore, upon substituting (5.8) into (3.6) we obtain a representation for all the solutions to the two-sided interpolation problem without constraints which is presented in the following:

THEOREM 5.2. *An operator H is a solution to the two-sided interpolation problem without constraints if and only if it admits the following representation*:

$$H = H_R + \hat{H}_L\Theta_R + \Theta_L h\Theta_R, \tag{5.10}$$

where H_R, $\hat{H}_L$, Θ_R, and Θ_L are given by (3.2), (4.1), (3.4), *and* (4.2), *respectively*.

In much the same way as it was done in the previous two sections we obtain a representation for all the solutions to the two-sided interpolation problem with normal constraints:

THEOREM 5.3. *There exists a solution of the two-sided interpolation problem of norm less than* 1 *if and only if a minimal norm solution*

$$H_{\min} = H_R + \hat{H}_L\Theta_R,$$

where H_R and $\hat{H}_L$ are given by (3.2) *and* (4.1), *is of norm less than* 1:

$$\|H_{\min}\|_2^2 = \|\hat{H}_L\|_2^2 + \|H_R\|_2^2 \le 1.$$

Moreover, all the solutions of norm less than 1 *can be described via formula* (5.10), *where in this case the parameter $h \in \mathscr{L}_2^{m\times n}$ satisfies the following norm constraint*:

$$\|h\|_2 \le \left(1 - \|H_R\|_2^2 - \|\hat{H}_L\|_2^2\right)^{1/2}.$$

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